Abstract

The initial value problem of an integrable system, such as the Nonlinear Schrödinger equation, is solved by subjecting the linear eigenvalue problem arising from its Lax pair to inverse scattering, and, thus, transforming it to a matrix Riemann-Hilbert problem (RHP) in the spectral variable. In the semiclassical limit, the method of nonlinear steepest descent ([4], [5]), supplemented by the g-function mechanism ([3]), is applied to this RHP to produce explicit asymptotic solution formulae for the integrable system. These formule are based on a hyperelliptic Riemann surface $\mathcal{R} = \mathcal{R}(x, t)$ in the spectral variable, where the space-time variables (x, t) play the role of external parameters. The curves in the x, t plane, separating regions of different genuses of $\mathcal{R}(x, t)$, are called breaking curves or nonlinear caustics. The genus of $\mathcal{R}(x, t)$ is related to the number of oscillatory phases in the asymptotic solution of the integrable system at the point x, t. An evolution theorem ([9]) guarantees the continuous evolution of the asymptotic solution in space-time away from the breaking curves.

In the case of the analytic scattering data $f(z; x, t)$ (in the NLS case, f is a normalized logarithm of the reflection coefficient with time evolution included), the primary role in the breaking mechanism is played by a phase function $\Im h(z; x, t)$, which is closely related to the g function. Namely, a break can be caused ([9]) either through the change of topology of zero level curves of $\Im h(z; x, t)$ (regular break), or through the interaction of zero level curves of $\Im h(z; x, t)$ with singularities of f (singular break). Every time a breaking curve in the x, t plane is reached, one has to prove the validity of the nonlinear steepest descent asymptotics in a region across the curve.

In this paper we prove that in the case of a regular break, the nonlinear steepest descent asymptotics can be "automatically" continued through the breaking curve (however, the expressions for the asymptotic solution will be different on the different sides of the curve). Our proof is based on the determinantal formula for $h(z; x, t)$ and its space and time derivatives, obtained in [7], [8]. Although the results are stated and proven for the focusing NLS equation, it is clear ([8]) that they can be reformulated for AKNS systems, as well as for the nonlinear steepest descend method in a more general setting.

Nonlinear steepest descent asymptotics for semiclassical limit of integrable systems: Continuation in the parameter space

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1 Introduction

The nonlinear steepest descent method, introduced in [4], [5], and its extension through the g-function mechanism introduced in [3], is widely used for asymptotic analysis of matrix Riemann-Hilbert problems (RHPs) with analytic jump matrices (that depend on additional parameters). Remarkable recent success stories of this method in such diverse areas as integrable systems, orthogonal polynomials, random matrices, approximation theory, etc., can be found, for example, in [1]. Let one of the additional parameters in the jump matrix, we denote it ε, be a small (semiclassical) parameter of the RHP. All the other parameters are called external parameters; particular external parameters considered in this paper are x, t, which have the meaning of space and time variables for the NLS equation. The g-function mechanism, when applicable, can be viewed as a way of calculating the leading order term of the ε asymptotics to the solution of a matrix RHP; it consists of reducing the matrix RHP to a scalar, independent of ε (but dependent on x, t) RHP (2) for the unknown function $g(z) = g(z; x, t)$, which is also a subject of additional requirements: modulation equations (4) and sign distributions (5). There is an underlying hyperelliptic Riemann surface $\mathcal{R} = \mathcal{R}(x, t)$, associated with $g(z; x, t)$; by the genus of $g(z; x, t)$, as well as the genus of the corresponding matrix RHP, we understand the genus of $\mathcal{R}(x, t)$. The genus of $g(z; x, t)$, in general, depends on external parameters x, t; a point x, t, where the genus of g undergoes a change, is called a breaking point. A curve consisting of breaking points is called *breaking curve* or *nonlinear caustics*. Conditions (4)-(5) with a certain genus N, which are valid on one side of the breaking curve, give no apriori guarantee that the same conditions with a new value of the genus will be valid on the other side. In particular, sign distributions (5) have to be established anew each time the breaking curve is crossed. For example, it took a lot of efforts to prove the transition from the genus zero to the genus two region, see Sect. 6.2 of

[9] and the corresponding part of [6]. Roughly speaking, the key result of the present paper is that sign distributions (5) with the properly chosen genus can be *automatically exteneded across a breaking curve*, provided that the change of genus (break) is *regular*, i.e., that the jump function of the scalar RHP (2) is analytic on the contour of this RHP, see details below.

The results of this paper are formulated for our model example, which is the matrix RHP that solves the inverse scattering problem for the focusing NLS

$$i\epsilon q_t + (\epsilon^2/2)q_{xx} + |q|^2 q = 0 \tag{1}$$

with decaying initial data $q(x, 0; \varepsilon)$ in the semiclassical limit $\varepsilon \to 0$. The contour and the jump matrix of this RHP and, accordingly, the contour and the jump function of the corresponding scalar RHP for g, are Schwarz symmetrical (see, for example, [9], Sect. 2,1, 2.4). However, it is an easy observation that our results do not depend on this symmetry and are applicable in a generic situation, for example, to the semiclassical limit of AKNS systems. A more detailed description of g-function is given below.

Let γ be a Schwarz-symmetrical oriented contour in $\mathbb{C}$ and $f_0(z)$ be a Schwarz-symmetrical analytic function in some domain of $\mathbb{C}$. We allow $f_0(z)$ to have a purely imaginary jump on the real axis. For simplicity, we assume γ to be a simple, smooth (except for a finitely many points) contour without self-intersections; moreover, we assume that $\gamma \cap \mathbb{R}$ consists of one and only one point μ. Let γ consists of $2n+1$, $n \in \mathbb{N}$, *main arcs* $\gamma_{m,j}$, $j = -n, -n-1, \cdots, n-1, n$, interlaced with $2n$ *complementary arcs* $\gamma_{c,j}$, $j = \pm 1, \pm 2, \cdots, \pm n$, see Figure 1, and let $\mu \in \gamma_{m,0}$. The main arcs can be considered as branchcuts of a hyperlliptic Riemann surface $\mathcal{R}$ of genus $N = 2n$ that lies at the core of the problem. The endpoints of main arcs are called branchpoints. Branchpoints located in the upper half-plane are denoted $\alpha_0, \alpha_2, \cdots, \alpha_{4n}$ respectively as we traverse γ in the direction of its orientation.

Because of the Schwarz symmetry of the problem, main arcs $\gamma_{m,j}$ and $\gamma_{m,-j}$, as well as complementary arcs $\gamma_{c,j}$ and $\gamma_{c,-j}$, are Schwarz symmetrical (but their orientation is antisymmetrical) for all the corresponding js. Unless specified otherwise, we use notations $\gamma_{m,j}$, $\gamma_{c,j}$ to denote the union of $\gamma_{m,j}$ and $\gamma_{m,-j}$ and the union of $\gamma_{c,j}$ and $\gamma_{c,-j}$, together with their orientations, respectively. It is clear that branchpoints in the lower half-plane are complex conjugates of the corresponding branchpoints α_{2j}, $j = 0, 2, \cdots, 2n$. We denote them $\alpha_{2j+1} = \overline{\alpha_{2j}}$.

The complex valued scalar g-function satisfies the following Riemann-Hilbert jump and analyticity conditions:

$$\begin{aligned} g_+ + g_- &= f + W_j \quad \text{on the main arc } \gamma_{m,j}, \ j = 0, \cdots, n \\ g_+ - g_- &= \Omega_j \quad \text{on the complementary arc } \gamma_{c,j}, \ j = 1, \cdots, n \\ g(z) &\text{ is analytic in } \bar{\mathbb{C}} \setminus \gamma, \end{aligned}$$

$$\tag{2}$$

where the function

$$f(z) = f_0(z) - zx - 2tz^2 \tag{3}$$

is a given input to the problem and all W_j and Ω_j are real constants. Furthermore, the g-function is required to have the following behavior at the branchpoints,

$$g(z) = O(z - \alpha_j)^{\frac{3}{2}} + \text{analytic function in a vicinity of } \alpha_j, \quad j = 0, 1, \cdots, 2N+1, \tag{4}$$
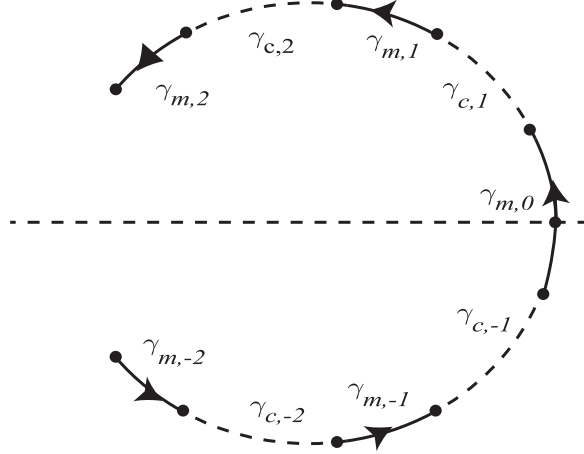
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Figure 1: Main and complementary arcs with $n = 2$.

which imposes $2N+2$ constraints, also known as *modulation equations* on the $2N+2$ branch-points, where $N = 2n$. All the branchpoints and all the real constants W_j and Ω_j are to be determined (through (2)-(4)). The only given data are the number $N + 1$ of branchcuts (or the genus N of the Riemann surface $\mathcal{R}$) and the function $f_0(z) = \frac{i}{2\varepsilon} \ln r(z)$, with x, t being the external parameters (space and time). Here $r(z)$ is the reflection coefficient for some initial data of (1).

Solution $g(z)$ of the RHP problem (2), which also satisfies modulation equations (4), is often known as the g-function of the nonlinear steepest descent method (in some papers, derivative $g'(z)$ is called the g-function). However, in order for the nonlinear steepest descent asymptotics to work (see, for example, [10]), the phase function $h = 2g - f$ should satisfy the following sign distribution inequalities:

$$\Im h < 0 \quad \text{on both sides of each main arc } \gamma_{m,j}, \ j = 0, 1, \cdots, n,$$
$$\Im h > 0 \quad \text{on at least one side of each complementary arc } \gamma_{c,j}, \ j = 1, \cdots, n.$$
$$(5)$$

These inequalities show that all the main arcs lie on zero level curves of $\Im h(z)$ and, unless prevented by singularities of $f_0(z)$, all the complementary arcs could be continuously deformed so that they also lie on zero level curves of $\Im h(z)$ (it is possible that parts of some complementary arcs would lie on $\mathbb{R}$). As we continuously deform external parameters x, t, the branchpoints α_j move according to (4), pulling (deforming) main and complemenary arcs of the contour $\gamma = \gamma(x, t)$ with them. We say that *the nonlinear steepest descent asymptotics is valid for some values of x, t if there exists $n \in \mathbb{N}$, such that all the branchpoints α_j stay away from $\mathbb{R} \cup \infty$ and the solution $g(z; x, t)$ of (2) satisfies (4) and (5).* If the nonlinear steepest descent asymptotics is valid for some x, t, then the expression for the leading order term (as $\varepsilon \to 0$) of the solution $q(x, t, \varepsilon)$ to the NLS (1) at x, t that corresponds to the scattering data $r(z)$ is given in [9], Main Theorem.

Suppose that the nonlinear steepest descent asymptotics is valid for some particular value of x_*, t_*. Then, according to the Evolution Theorem (Theorem 3.2) of [9], $g(z; x, t)$ with the same genus $N = 2n$ satisfies (4) and (5) in a neighborhood of x_*, t_* of the x, t-plane. If x, t

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are evolving further (outside this neighborhood) along some piecewise-smooth curve Σ in the x, t-plane, $x_*, t_* \in \Sigma$, then it is possible ([9], Section 3) that an inequality of (5) fails at a point $x_b, t_b \in \Sigma$ (breaking point). This failure can be caused by one of the following two reasons: a) regular, when a change of the topology of zero level curves of $\Im h(z) = \Im h(z; x, t)$ at $(x, t) = (x_b, t_b)$ affects contour γ; b) singular, when the contour $\gamma = \gamma(x, t)$ interacts (collides or encircles) with singularities (including branchcuts) of $f_0(z)$ at $(x, t) = (x_b, t_b)$.

The goal of this paper is to address the regular breaking (scenario a)), leaving the case of the singular breaking (scenario b)) to be addressed elsewhere. Let the genus of $g(z; x_*, t_*)$ be $N = 2n$. According to [9], Section 3, the change of topology of zero level curves of $\Im h(z)$ at the breaking point x_b, t_b contains two generic possibilities: i) two branches of zero level curve of $\Im h(z)$ collide at some point $z_0 \in \gamma$ that is not a branchpoint; ii) two adjacent branchpoints collide at some point z_0 (collision of nonadjacent branchpoints creates a loop that encircles some singularities). In any case, z_0 is called a breaking point in the spectral plane that corresponds to the breaking point x_b, t_b in the x, t plane. In the case i) we can plant a pair of branchpoints at the breaking point z_0 and another pair of branchpoints at the conjugated breaking point $\bar{z}_0$. That allows us to consider the corresponding hyperelliptic surface $\mathcal{R} = \mathcal{R}(x, t)$ as having genus N at the breaking point x_b, t_b before planting the branchpoints and, simultaneously, as having genus $N + 2$ after the planting. As we evolve further along Σ, a new pair of main arcs (if $z_0 \in \gamma_c$) or of complementary arcs (if $z_0 \in \gamma_m$) with endpoints evolving from z_0 and from $\bar{z}_0$ opens up. The case ii) can be described by evolving along Σ through the breaking point x_b, t_b in the opposite direction. By removing a pair of colliding branchpoints (and their conjugates), we reduce the genus of $\mathcal{R}$ by two, say, from N to $N - 2$. In degenerate cases, several zero level curves of $\Im h(z)$ meet at the same point z_0, which may or may not be a branchpoint. Then

$$h(z; x_b, t_b) = C + O(z - z_0)^m \ , \tag{6}$$

where $2m \in \mathbb{Z}^+$ and C is a real constant. m is called the degree of degenerate breaking point z_0. Note that if the breaking point z_0 is also a branchpoint, then m is a half-integer number, otherwise, m is an integer. The number of zero level curves of $\Im h(z)$, emanating from z_0, is $2m$, and the number of the branchpoints, "born" at the breaking point z_0, is $2m - 2$. For example, two branchpoints emanate from z_0 of degree two (called a double point), three branchpoints emanate from z_0 of degree $5/2$ (called a triple point), etc. In [9], the only triple point was the point at the tip (corner) of the breaking curve; it was the point where the very first break (in the process of time evolution) occurs. It is possible that there are several breaking points in the spectral plane (without counting complex conjugated points) that correspond to the same breaking point x_b, t_b, for example, when several inequalities of (5) fail at x_b, t_b. Such breaking points x_b, t_b are degenerate (non-generic). It is shown in Sect. 4 that degenerate breaking points are isolated points in the x, t-plane.

Let $g^{(N)}(z)$ denote the solution of the RHP (2) with $N + 1 = 2n + 1$ main arcs, i.e., $g^{(N)}(z)$ denotes a g function of the genus N, and let $h^{(N)}(z) = 2g^{(N)}(z) - f(z)$. The Degeneracy Theorem (Theorem 3.1) of [9] states that $h^{(N+2)}(z; x_b, t_b) \equiv h^{(N)}(z; x_b, t_b)$, provided that x_b, t_b is a regular breaking point. The Degeneracy Theorem is an important tool in tracking the signs of $\Im h(z; x, t)$, and with them, the validity of of the nonlinear steepest descent asymptotics, through breaking points. However, it does not guarantee the correct sign distribution, i.e., inequalities (5), past the breaking point, i.e., in the genus $N + 2$ or in the

genus $N - 2$ regions. For example, in the case i) it does not guarantee that the signs of $\Im h$ around the newborn arc are correct, i.e., that the corresponding inequality from (5) is satisfied (signs around all the other arcs are correct by the continuity argument). To track the signs of $\Im h(z)$ through the breaking point, it would be very helpful to establish that not only $h^{(N+2)}(z)$ and $h^{(N)}(z)$ are equivalent at the breaking point, but that so are their partial derivatives with respect to external parameters, i.e., h_x and h_t. The latter statements do not follow from the Degeneracy Theorem directly, since $h^{(N+2)}(z; x_b, t_b) \equiv h^{(N)}(z; x_b, t_b)$ only at the breaking point x_b, t_b, but not in any vicinity of this point.

The key observation of this paper is that, in fact,

$$h_x^{(N+2)}(z; x_b, t_b) \equiv h_x^{(N)}(z; x_b, t_b) \quad \text{and} \quad h_t^{(N+2)}(z; x_b, t_b) \equiv h_t^{(N)}(z; x_b, t_b) \tag{7}$$

at any regular and generic breaking point x_b, t_b. The proof of (7) involves the determinant formula from [8]. Equations (7) allow us to prove that the nonlinear steepest descent asymptotics *is always preserved* when one passes through a regular and generic breaking point, provided that the genus of the problem is adjusted accordingly. Speaking somewhat lousely, we can formulate the following *regular continuation principle.*

Regular continuation principle for the nonlinear steepest descent asymptotics: Let the nonlinear steepest descent asymptotics for solution $q(x, t, \varepsilon)$ of the NLS (1) be valid at some point (x_b, t_b). If (x_*, t_*) is an arbitrary point, connected with (x_b, t_b) by a piecewise-smooth path Σ, if the countour $\gamma(x, t)$ of the RHP (2) does not interact with singularities of $f_0(z)$ as (x, t) varies from (x_b, t_b) to (x_*, t_*) along Σ, and if all the branchpoints are bounded and stay away from the real axis, then the nonlinear steepest descent asymptotics (with the proper choice of the genus) is also valid at (x_*, t_*).

This principle will be proved in Section 4. Some important facts about the determinantal formula are provided in Section 2, whereas formula (7) is proven in Theorem 3.1, Section 3.

2   Determinantal formula

Theorem 3.1, which is the central part of the regular continuation principle, is also an important advancement of the Degeneracy Theorem from [9]. Its proof is based on the determinant representation of h and its immediate consequences, obtained in [7], [8]. Some basic facts from [8] are given in this section.

Assuming that $W_0, W_j, \Omega_j, j = 1, 2, \cdots, n$, and $\alpha_j, j = 0, 1, \cdots, 4n + 1$, are known, the solution to the RHP (2) is given by

$$g(z) = \frac{R(z)}{2\pi i} \left[\int_\gamma \frac{f(\zeta)}{(\zeta - z)R(\zeta)_+} d\zeta + \sum_{j=0}^n \int_{\gamma_{m,j}} \frac{W_j}{(\zeta - z)R(\zeta)_+} d\zeta + \sum_{j=1}^n \int_{\gamma_{c,j}} \frac{\Omega_j}{(\zeta - z)R(\zeta)} d\zeta \right]. \tag{8}$$

where the radical $R(z) = \sqrt{\prod_{j=0}^{4n+1}(z - \alpha_j)}$ has branchcuts $\gamma_{m,j}, j = 0, 1, \cdots, n$, i.e., $\mathcal{R}$ is the Riemann surface (of the genus $N = 2n$) of the radical $R(z)$. We fix the branch of $R(z)$ by the requirement

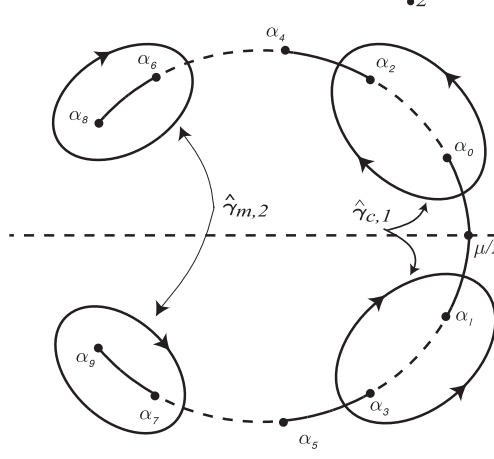$$\lim_{z \to \infty} \frac{R(z)}{z^{N+1}} = -1 \tag{9}$$

Figure 2: Contours $\hat{\gamma}_{m,2}$, $\hat{\gamma}_{c,1}$

on the main sheet of $\mathcal{R}$.

Expressing the integrals over main and complementary arcs as integrals over the loops shown in Fig. 2, i.e., as α cycles and as combinations of β cycles of the hyperelliptic surface $\mathcal{R}$, we obtain

$$g(z) = \frac{R(z)}{4\pi i} \left[\oint_{\hat{\gamma}} \frac{f(\zeta)}{(\zeta - z)R(\zeta)} d\zeta + \sum_{j=0}^{n} \oint_{\hat{\gamma}_{m,j}} \frac{W_j}{(\zeta - z)R(\zeta)} d\zeta + \sum_{j=1}^{n} \oint_{\hat{\gamma}_{c,j}} \frac{\Omega_j}{(\zeta - z)R(\zeta)} d\zeta \right],$$
(10)

where the loops $\hat{\gamma}_{m,j}$ around main arcs $\gamma_{m,j}$ have negative (clock-wise) orientation (an α cycle) and the loops $\hat{\gamma}_{c,j}$ around complementary arcs $\gamma_{c,j}$ have positive (counterclock-wise) orientation. Here the part of $\hat{\gamma}_{c,j}$ on the main sheet of $\mathcal{R}$ has the same orientation as $\gamma_{c,j}$ and the part of $\hat{\gamma}_{c,j}$ on the secondary sheet of $\mathcal{R}$ has the opposite orientation (a β cycle). Alternatively, $\hat{\gamma}_{c,j}$ can be considered as a union of two arcs on the main sheet of $\mathcal{R}$ surrounding $\gamma_{c,j}$ with opposite orientations. The loop $\hat{\gamma}$ is a negatively oriented contour surrounding γ. All loops are contained in S and are contractible to their corresponding arcs without passing through z (that mean that the loops are pinched to their respective contours at the points of nonanalyticity of $f_0(z)$)..

Deforming $\hat{\gamma}$ so that z *becomes inside the loop* $\hat{\gamma}$ and still outside the loops $\hat{\gamma}_{m,j}$ and $\hat{\gamma}_{c,j}$, we obtain

$$h(z) = \frac{R(z)}{2\pi i} \left[\oint_{\hat{\gamma}} \frac{f(\zeta)}{(\zeta - z)R(\zeta)} d\zeta + \sum_{j=0}^{n} \oint_{\hat{\gamma}_{m,j}} \frac{W_j}{(\zeta - z)R(\zeta)} d\zeta + \sum_{j=1}^{n} \oint_{\hat{\gamma}_{c,j}} \frac{\Omega_j}{(\zeta - z)R(\zeta)} d\zeta \right],$$
(11)

where

$$h(z) = 2g(z) - f(z).$$
(12)

The function $h(z)$ is obtained by multiplying $g(z)$ by a factor of 2 and the residue $-f$ being picked up as z cuts through the loop $\hat{\gamma}$.

According to (9) and (10), $g(z) \sim O(z^N)$ as $z \to \infty$. Without any loss of generality, we can assume that $W_0 = 0$ (otherwise, replacing the solution $g(z)$ of (2) by $g(z) - \frac{1}{2}W_0$, we

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add $-W_0$ to jump constant W_j on every main arc $\gamma_{m,j}$, as well as to $g(\infty)$, without changing any of the jump constants Ω_j). The requirement that $g(z)$ is analytic at $z = \infty$, see (2), together with the Schwarz symmetry define the system of $N = 2n$ real linear equations

$$\oint_{\hat{\gamma}} \frac{\zeta^k f(\zeta)}{R(\zeta)} d\zeta + \sum_{j=1}^{N} \oint_{\hat{\gamma}_{m,j}} \frac{W_j \zeta^k}{R(\zeta)} d\zeta + \sum_{j=1}^{N} \oint_{\hat{\gamma}_{c,j}} \frac{\Omega_j \zeta^k}{R(\zeta)} d\zeta = 0, \qquad k = 0, 1, \cdots, N-1, \quad (13)$$

for N real variables $W_j, \Omega_j, \ j = 1, 2, \cdots, n$. Let us introduce

$$D = \begin{vmatrix} \oint_{\hat{\gamma}_{m,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,1}} \frac{\zeta^{2n-1} d\zeta}{R(\zeta)} \\ \cdots & \cdots & \cdots \\ \oint_{\hat{\gamma}_{m,n}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,n}} \frac{\zeta^{2n-1} d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}_{c,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,1}} \frac{\zeta^{2n-1} d\zeta}{R(\zeta)} \\ \cdots & \cdots & \cdots \\ \oint_{\hat{\gamma}_{c,n}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,n}} \frac{\zeta^{N-1} d\zeta}{R(\zeta)} \end{vmatrix} \qquad (14)$$

and

$$K(z) = \frac{1}{2\pi i} \times \begin{vmatrix} \oint_{\hat{\gamma}_{m,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,1}} \frac{\zeta^{2n-1} d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,1}} \frac{d\zeta}{(\zeta-z)R(\zeta)} \\ \cdots & \cdots & \cdots & \cdots \\ \oint_{\hat{\gamma}_{m,n}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,n}} \frac{\zeta^{2n-1} d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,n}} \frac{d\zeta}{(\zeta-z)R(\zeta)} \\ \oint_{\hat{\gamma}_{c,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,1}} \frac{\zeta^{2n-1} d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,1}} \frac{d\zeta}{(\zeta-z)R(\zeta)} \\ \cdots & \cdots & \cdots & \cdots \\ \oint_{\hat{\gamma}_{c,n}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,n}} \frac{\zeta^{2n-1} d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,n}} \frac{d\zeta}{(\zeta-z)R(\zeta)} \\ \oint_{\hat{\gamma}} \frac{f(\zeta) d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}} \frac{\zeta^{2n-1} f(\zeta) d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}} \frac{f(\zeta) d\zeta}{(\zeta-z)R(\zeta)} \end{vmatrix} . \qquad (15)$$

Note that D can be reduced to the determinant made of basic holomorphic differentials of $\mathcal{R}$ ([8]) and thus $D \neq 0$. The latter implies solvability of (13) with any $f(z) = f(z; x, t)$ given by (3). That allows us to obtain

$$h(z) = \frac{R(z)}{D} K(z) , \qquad (16)$$

where z is inside the loop $\hat{\gamma}$ but outside all other loops $\hat{\gamma}_{m,j}, \hat{\gamma}_{c,j}$. Assumption that z is outside the loop $\hat{\gamma}$ yields

$$g(z) = \frac{R(z)}{2D} K(z) . \qquad (17)$$

Equation (16) allows us ([8]) to obtain

$$\frac{d}{dx} h(z) = \frac{R(z)}{D} \frac{\partial}{\partial x} K(z), \qquad \frac{d}{dt} h(z) = \frac{R(z)}{D} \frac{\partial}{\partial t} K(z) . \qquad (18)$$

Combining (18) with (15) and (3), one can easily obtain

$$\frac{\partial}{\partial x}K(z) = \begin{vmatrix} \oint_{\hat{\gamma}_{m,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,1}} \frac{\zeta^{2n-2}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,1}} \frac{d\zeta}{(\zeta-z)R(\zeta)} \\ \cdots & \cdots & \cdots & \cdots \\ \oint_{\hat{\gamma}_{m,n}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,n}} \frac{\zeta^{2n-2}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,n}} \frac{d\zeta}{(\zeta-z)R(\zeta)} \\ \oint_{\hat{\gamma}_{c,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,1}} \frac{\zeta^{2n-2}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,1}} \frac{d\zeta}{(\zeta-z)R(\zeta)} \\ \cdots & \cdots & \cdots & \cdots \\ \oint_{\hat{\gamma}_{c,n}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,n}} \frac{\zeta^{2n-2}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,n}} \frac{d\zeta}{(\zeta-z)R(\zeta)} \end{vmatrix} \tag{19}$$

and

$$\frac{\partial}{\partial t}K(z) = -2 \begin{vmatrix} \oint_{\hat{\gamma}_{m,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,1}} \frac{\zeta^{2n-3}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,1}} \frac{d\zeta}{(\zeta-z)R(\zeta)} & \oint_{\hat{\gamma}_{m,1}} \frac{\zeta^{2n-1}d\zeta}{R(\zeta)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \oint_{\hat{\gamma}_{m,n}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,n}} \frac{\zeta^{2n-3}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,n}} \frac{d\zeta}{(\zeta-z)R(\zeta)} & \oint_{\hat{\gamma}_{m,n}} \frac{\zeta^{2n-1}d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}_{c,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,1}} \frac{\zeta^{2n-3}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,1}} \frac{d\zeta}{(\zeta-z)R(\zeta)} & \oint_{\hat{\gamma}_{c,1}} \frac{\zeta^{2n-1}d\zeta}{R(\zeta)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \oint_{\hat{\gamma}_{c,n}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,n}} \frac{\zeta^{2n-3}d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,n}} \frac{d\zeta}{(\zeta-z)R(\zeta)} & \oint_{\hat{\gamma}_{c,n}} \frac{\zeta^{2n-1}d\zeta}{R(\zeta)} \end{vmatrix} + \sum_{j=0}^{4n+1} \alpha_j \frac{\partial}{\partial x}K(z) \; . \tag{20}$$

Equations (14)-(20) can be, in fact, extended to a more general situation, where $f_0(z)$ and contour γ are not necessarily Schwarz-symmetrical (this would extend the nonlinear steepest descent mehtod from the NLS to some general AKNS systems). In particular (see [8]):

$$D = \begin{vmatrix} \oint_{\Gamma_{m,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\Gamma_{m,1}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \overline{\oint_{\Gamma_{m,1}} \frac{d\zeta}{R(\zeta)}} & \cdots & \overline{\oint_{\Gamma_{m,1}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \oint_{\Gamma_{m,N}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\Gamma_{m,N}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \overline{\oint_{\Gamma_{m,N}} \frac{d\zeta}{R(\zeta)}} & \cdots & \overline{\oint_{\Gamma_{m,N}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)}} \\ \oint_{\Gamma_{c,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\Gamma_{c,1}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \overline{\oint_{\Gamma_{c,1}} \frac{d\zeta}{R(\zeta)}} & \cdots & \overline{\oint_{\Gamma_{c,1}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \oint_{\Gamma_{c,N}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\Gamma_{c,N}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \overline{\oint_{\Gamma_{c,N}} \frac{d\zeta}{R(\zeta)}} & \cdots & \overline{\oint_{\Gamma_{c,N}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)}} \end{vmatrix} \tag{21}$$

and

$$K(z) = \frac{1}{2\pi i} \begin{vmatrix} \oint_{\Gamma_{m,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\Gamma_{m,1}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \overline{\oint_{\Gamma_{m,1}} \frac{d\zeta}{R(\zeta)}} & \cdots & \overline{\oint_{\Gamma_{m,1}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)}} & \oint_{\Gamma_{m,1}} \frac{d\zeta}{(\zeta-z)R(\zeta)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \oint_{\Gamma_{m,N}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\Gamma_{m,N}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \overline{\oint_{\Gamma_{m,N}} \frac{d\zeta}{R(\zeta)}} & \cdots & \overline{\oint_{\Gamma_{m,N}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)}} & \oint_{\Gamma_{m,N}} \frac{d\zeta}{(\zeta-z)R(\zeta)} \\ \oint_{\Gamma_{c,1}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\Gamma_{c,1}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \overline{\oint_{\Gamma_{c,1}} \frac{d\zeta}{R(\zeta)}} & \cdots & \overline{\oint_{\Gamma_{c,1}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)}} & \oint_{\Gamma_{c,1}} \frac{d\zeta}{(\zeta-z)R(\zeta)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \oint_{\Gamma_{c,N}} \frac{d\zeta}{R(\zeta)} & \cdots & \oint_{\Gamma_{c,N}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)} & \overline{\oint_{\Gamma_{c,N}} \frac{d\zeta}{R(\zeta)}} & \cdots & \overline{\oint_{\Gamma_{c,N}} \frac{\zeta^{N-1}d\zeta}{R(\zeta)}} & \oint_{\Gamma_{c,N}} \frac{d\zeta}{(\zeta-z)R(\zeta)} \\ \oint_{\hat{\gamma}} \frac{f(\zeta)d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}} \frac{\zeta^{N-1}f(\zeta)d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}} \frac{f(\zeta)d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}} \frac{\zeta^{N-1}f(\zeta)d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}} \frac{f(\zeta)d\zeta}{(\zeta-z)R(\zeta)} \end{vmatrix} , \tag{22}$$

where $\Gamma_{m,j}$, $j = 1,2,\cdots,N$, and $\Gamma_{c,j}$, $j = 1,2,\cdots,N$, denote basic α and β cicles of the corresponding hyperelliptic surface. In the case of our countour γ, see Fig. 1, $\Gamma_{m,j}$,

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$j = 1, 2, \cdots, n$, is $\hat{\gamma}^+_{m,j}$, $\Gamma_{m,j}$, $j = n+1, n+2, \cdots, 2n$, is $\hat{\gamma}^-_{m,j}$, where $\lambda^\pm$ denote parts of the contour λ that lie the upper and lower halfplanes respectively. Similarly, $\Gamma_{c,j}$, $j = 1, 2, \cdots, n$, is $\hat{\gamma}^+_{c,j}$, $\Gamma_{c,j}$, $j = n+1, n+2, \cdots, 2n$, is $\hat{\gamma}^-_{c,j}$.

According to [8], $D \neq 0$ and (16)-(18) are still valid when D and K are given by (21), (22). Denoting by K_j, $j = 1, 2 \cdots, N$, the jth column in (21), and by $Q(z)$ the first $2n$ entries in the last column of (22), we can easily obtain

$$
\begin{aligned}
\frac{\partial}{\partial x} K(z) = & -\det(K_1, \cdots, K_{N-1}, \overline{K_1}, \cdots, \overline{K_{N-1}}, K_N + \overline{K_N}, Q(z)) \\
\frac{\partial}{\partial t} K(z) = & 2 \det(K_1, \cdots, K_{N-2}, K_N, \overline{K_1}, \cdots, \overline{K_{N-2}}, K_{N-1} + \overline{K_{N-1}}, \overline{K_N}, Q(z)) \\
& + \sum_{j=0}^{4n+1} \alpha_j \frac{\partial}{\partial x} K(z) .
\end{aligned}
$$

$$(23)$$

3　Continuity of h_x and h_t across a breaking curve

Theorem 3.1. *Let (x_b, t_b) be a regular breaking point and α be the corresponding breaking point in the spectral plane that is a double point. Let $(x_b, t_b) \in l$, where l is a breaking curve that separates regions of genus $2n$ and of genus $2n - 2$, $n \in \mathbb{Z}^+$. If $h^{(2n)}(z; x, t)$ denotes the function h in the genus $2n$ region (on one side of l) and $h^{(2n-2)}(z; x, t)$ denote the function h in the genus $2n - 2$ region (on the other side l), then at the point $(x, t) = (x_b, t_b)$ we have*

$$
\frac{d}{dx} h^{(2n-2)}(z; x, t) \equiv \frac{d}{dx} h^{(2n)}(z; x, t) \quad \text{and} \quad \frac{d}{dx} h^{(2n-2)}(z; x, t) \equiv \frac{d}{dx} h^{(2n)}(z; x, t). \quad (24)
$$

Proof. The proof is based on formulae (18). We consider the situation when the pair of main arcs $\gamma_{m,n}$ collapses into a pair of double points α and $\bar{\alpha}$. That means that the corresponding branchpoints α_{4n-2} and α_{4n} are collapsing into a point α and the their complex-conjugated branchpoints $\alpha_{4n-1}, \alpha_{4n+1}$ are collapsing into $\bar{\alpha}$. It is convenient to introduce $\delta = |\alpha_{4n-2}(x, t) - \alpha_{4n}(x, t)|$, where $\delta \to 0$.

We first evaluate the 2×2 determinant D_2, given by (14), with $n = 1$ in the limit $\delta \to 0$. Observe that

$$
D_2 = \begin{vmatrix} \oint_{\hat{\gamma}_m} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_m} \frac{\zeta d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}_c} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_c} \frac{\zeta d\zeta}{R(\zeta)} \end{vmatrix} = \begin{vmatrix} \oint_{\hat{\gamma}_m} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_m} \frac{(\zeta - \alpha_2)d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}_c^+} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_c^+} \frac{(\zeta - \alpha_2)d\zeta}{R(\zeta)} \end{vmatrix} + \begin{vmatrix} \oint_{\hat{\gamma}_m} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_m} \frac{(\zeta - \bar{\alpha}_2)d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}_c^-} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_c^-} \frac{(\zeta - \bar{\alpha}_2)d\zeta}{R(\zeta)} \end{vmatrix} = D_2^+ + D_2^- ,
$$

$$(25)$$

where $\nu^\pm$ denote parts of the contour ν in the upper/lower half-planes respectively. Here we use notation γ_m for $\gamma_{m,1}$ and γ_c for $\gamma_{c,1}$. It is clear that all but $(2, 1)$ entries of both determinants D_2^+, D_2^- stay bounded as $\delta \to 0$. Using the fact that in the limit $\delta \to 0$

$$
R(z) = (z - \alpha)(z - \bar{\alpha}) R_0(z) + O(\delta) \tag{26}
$$

provided that z is separated from α and from $\bar{\alpha}$, where

$$
R_0(z) = \sqrt{(z - \alpha_0)(z - \bar{\alpha}_0)} , \tag{27}
$$

we obtain

$$\oint_{\hat{\gamma}_m} \frac{(\zeta - \alpha_2)d\zeta}{R(\zeta)} = -\frac{2\pi i}{R_0(\bar{\alpha})}(1 + o(1)) \tag{28}$$

as $\delta \to 0$, where $\alpha = a + ib$. Using the similar estimate for $\oint_{\hat{\gamma}_m} \frac{(\zeta - \bar{\alpha}_2)d\zeta}{R(\zeta)}$, we finally arrive at

$$D_2 = \frac{2\pi i}{2ib|R_0(\alpha)|^2} \left[\int_{\gamma_c^+} \frac{d\zeta}{\sqrt{(\zeta - \alpha_4)(\zeta - \alpha_2)}} - \int_{\gamma_c^-} \frac{d\zeta}{\sqrt{(\zeta - \bar{\alpha}_4)(\zeta - \bar{\alpha}_2)}} \right] + O(1)$$

$$= \frac{2\pi}{b|R_0(\alpha)|^2} \ln \delta + O(1) \tag{29}$$

as $\delta \to 0$.

Consider now D_{2n}, given by (14) with $n = 2, 3, \cdots$, where the main arc $\gamma_{m,n}$ is collapsing into a point α when $(x, t) \to (x_b, t_b)$. Rewriting

$$D_{2n} = (-1)^{n-1} \begin{vmatrix} \oint_{\hat{\gamma}_{m,1}} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,1}} \frac{\zeta d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,1}} \frac{(\zeta-\alpha_*)(\zeta-\bar{\alpha}_*)d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,1}} \frac{\zeta^{2n-3}(\zeta-\alpha_*)(\zeta-\bar{\alpha}_*)d\zeta}{R(\zeta)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \oint_{\hat{\gamma}_{m,n-1}} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,n-1}} \frac{\zeta d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,n-1}} \frac{(\zeta-\alpha_*)(\zeta-\bar{\alpha}_*)d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,n-1}} \frac{\zeta^{2n-3}(\zeta-\alpha_*)(\zeta-\bar{\alpha}_*)d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}_{c,1}} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,1}} \frac{\zeta d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,1}} \frac{(\zeta-\alpha_*)(\zeta-\bar{\alpha}_*)d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,1}} \frac{\zeta^{2n-3}(\zeta-\alpha_*)(\zeta-\bar{\alpha}_*)d\zeta}{R(\zeta)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \oint_{\hat{\gamma}_{c,n-1}} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,n-1}} \frac{\zeta d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,n-1}} \frac{(\zeta-\alpha_*)(\zeta-\bar{\alpha}_*)d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,n-1}} \frac{\zeta^{2n-3}(\zeta-\alpha_*)(\zeta-\bar{\alpha}_*)d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}_{m,n}} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,n}} \frac{\zeta d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,n}} \frac{(\zeta-\alpha_*)(\zeta-\bar{\alpha}_*)d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,n}} \frac{\zeta^{2n-3}(\zeta-\alpha_*)(\zeta-\bar{\alpha}_*)d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}_{c,n}} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,n}} \frac{\zeta d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,n}} \frac{(\zeta-\alpha_*)(\zeta-\bar{\alpha}_*)d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,n}} \frac{\zeta^{2n-3}(\zeta-\alpha_*)(\zeta-\bar{\alpha}_*)d\zeta}{R(\zeta)} \end{vmatrix}, \tag{30}$$

where $\alpha_* = \alpha_{4n-2}$, and using (26), where

$$R_0(z) = \sqrt{\prod_{j=0}^{2n-2} (z - \alpha_{2j})(z - \overline{\alpha_{2j}})}, \tag{31}$$

we see that all but the first two entries of the $(2n - 1)$th (next to the last) row of D_{2n} are approaching zero as $\delta \to 0$. Taking into account (29) and the fact that all the entries $(2n, j)$, $j = 3, 4, \cdots, 2n$ of the determinant (30) are bounded, we obtain

$$D_{2n+2} = (-1)^{n-1} D_2 \left[D_{2n-2} + o(1) \right] \tag{32}$$

as $\delta \to 0$, where D_{2n-2} denotes the determinant built on the main arcs $\gamma_{m,1}, \cdots, \gamma_{m,n-1}$ and the corresponding complementary arcs.

Our next step is evaluation of $\frac{d}{dx} h^{(2n)}(z; x, t) = \frac{R(z)}{D_{2n}} \frac{\partial}{\partial x} K^{(2n)}(z; x, t)$ in the limit $\delta \to 0$, i.e., when $(x, t) \to (x_b, t_b)$. Here $K^{(2n)}(z) = K^{(2n)}(z; x, t)$ denotes $2n + 1$ dimensional determinant $K(z)$ given by (15). This evaluation is based on the identity

$$\frac{1}{(\zeta - z)(\zeta - \alpha_*)(\zeta - \bar{\alpha}_*)} = \frac{1}{(z - \alpha_*)(z - \bar{\alpha}_*)} \left[\frac{1}{\zeta - z} - \frac{\zeta + z - 2\Re\alpha_*}{(\zeta - \alpha_*)(\zeta - \bar{\alpha}_*)} \right], \tag{33}$$

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where $\alpha_* \in \mathbb{C}$ is arbitrary. Using (33), the integrand $\frac{1}{(\zeta-z)R(\zeta)}$ of the last column of determinant (15) can be represented as

$$\frac{1}{(\zeta-z)R(\zeta)} = \frac{(\zeta-\alpha_*)(\zeta-\bar\alpha_*)}{(z-\alpha_*)(z-\bar\alpha_*)(\zeta-z)R(\zeta)} - \frac{\zeta+z-2\Re\alpha_*}{(z-\alpha_*)(z-\bar\alpha_*)R(\zeta)} . \tag{34}$$

Since the latter term can be eliminated by linear operations with columns of (15), we obtain

$$\frac{\partial}{\partial x}K^{(2n)}(z) = \frac{(-1)^{n-1}}{(z-\alpha_*)(z-\bar\alpha_*)}\times$$

$$\left|\begin{array}{ccccccc}
\oint_{\hat\gamma_{m,1}}\frac{d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{m,1}}\frac{\zeta d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{m,1}}\frac{(\zeta-\alpha_*)(\zeta-\bar\alpha_*)d\zeta}{R(\zeta)} & \cdots & \oint_{\hat\gamma_{m,1}}\frac{\zeta^{2n-4}(\zeta-\alpha_*)(\zeta-\bar\alpha_*)d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{m,1}}\frac{(\zeta-\alpha_*)(\zeta-\bar\alpha_*)d\zeta}{(\zeta-z)R(\zeta)} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\oint_{\hat\gamma_{m,n-1}}\frac{d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{m,n-1}}\frac{\zeta d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{m,n-1}}\frac{(\zeta-\alpha_*)(\zeta-\bar\alpha_*)d\zeta}{R(\zeta)} & \cdots & \oint_{\hat\gamma_{m,n-1}}\frac{\zeta^{2n-4}(\zeta-\alpha_*)(\zeta-\bar\alpha_*)d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{m,n-1}}\frac{(\zeta-\alpha_*)(\zeta-\bar\alpha_*)d\zeta}{(\zeta-z)R(\zeta)} \\
\oint_{\hat\gamma_{c,1}}\frac{d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{c,1}}\frac{\zeta d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{c,1}}\frac{(\zeta-\alpha_*)(\zeta-\bar\alpha_*)d\zeta}{R(\zeta)} & \cdots & \oint_{\hat\gamma_{c,1}}\frac{\zeta^{2n-4}(\zeta-\alpha_*)(\zeta-\bar\alpha_*)d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{c,1}}\frac{(\zeta-\alpha_*)(\zeta-\bar\alpha_*)d\zeta}{(\zeta-z)R(\zeta)} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\oint_{\hat\gamma_{c,n-1}}\frac{d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{c,n-1}}\frac{\zeta d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{c,n-1}}\frac{(\zeta-\alpha_*)(\zeta-\bar\alpha_*)d\zeta}{R(\zeta)} & \cdots & \oint_{\hat\gamma_{c,n-1}}\frac{\zeta^{2n-4}(\zeta-\alpha_*)(\zeta-\bar\alpha_*)d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{c,n-1}}\frac{(\zeta-\alpha_*)(\zeta-\bar\alpha_*)d\zeta}{(\zeta-z)R(\zeta)} \\
\oint_{\hat\gamma_{m,n}}\frac{d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{m,n}}\frac{\zeta d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{m,n}}\frac{(\zeta-\alpha_*)(\zeta-\bar\alpha_*)d\zeta}{R(\zeta)} & \cdots & \oint_{\hat\gamma_{m,n}}\frac{\zeta^{2n-4}(\zeta-\alpha_*)(\zeta-\bar\alpha_*)d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{m,n}}\frac{(\zeta-\alpha_*)(\zeta-\bar\alpha_*)d\zeta}{(\zeta-z)R(\zeta)} \\
\oint_{\hat\gamma_{c,n}}\frac{d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{c,n}}\frac{\zeta d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{c,n}}\frac{(\zeta-\alpha_*)(\zeta-\bar\alpha_*)d\zeta}{R(\zeta)} & \cdots & \oint_{\hat\gamma_{c,n}}\frac{\zeta^{2n-4}(\zeta-\alpha_*)(\zeta-\bar\alpha_*)d\zeta}{R(\zeta)} & \oint_{\hat\gamma_{c,n}}\frac{(\zeta-\alpha_*)(\zeta-\bar\alpha_*)d\zeta}{(\zeta-z)R(\zeta)}
\end{array}\right| .$$
$$\tag{35}$$

Let $M^{(2n-2)}(z)$ denote the minor of (35) that consists of the first $2n-2$ rows and the last $2n-2$ columns. Choosing $\alpha_* = \alpha_{4n-2}$, we can replace the factor $\frac{(\zeta-\alpha_*)(\zeta-\bar\alpha_*)}{R(\zeta)}$ in all the integrands of the minor $M^{(2n-2)}(z)$ by $\frac{1}{R_0(\zeta)}$ with the accuracy $O(\delta)$ as $\delta \to 0$. So, $M^{(2n-2)}(z) = \frac{\partial}{\partial x}K^{(2n-2)}(z) + O(\delta)$. Note also that, for any fixed $z \neq \alpha$, all but the first two enties of the $(2n-1)$st row of (35) have the order $O(\delta)$, and all but the first two enties of the last row of (35) are bounded as $\delta \to 0$. Thus, applying to (35) the arguments of (30), we obtain

$$\frac{\partial}{\partial x}K^{(2n)}(z) = \frac{(-1)^{n-1}D_2\frac{\partial}{\partial x}K^{(2n-2)}(z)}{(z-\alpha_{4n-2})(z-\bar\alpha_{4n-2})} + O(\delta) \tag{36}$$

as $\delta \to 0$, which holds uniformly in z on compact subsets of $\mathbb{C}\setminus\{\alpha,\bar\alpha\}$. Now, according to (19), (32), (31) and (36), we have

$$\begin{aligned}
\frac{d}{dx}h^{(2n)}(z;x,t)\Big|_{(x,t)=(x_b,t_b)} &= \lim_{\delta\to 0}\frac{(-1)^{n-1}D_2 R(z)\frac{\partial}{\partial x}K^{(2n-2)}(z;x,t)}{(z-\alpha_{4n-2})(z-\bar\alpha_{4n-2})D_{2n}} \\
&= \frac{R_0(z)}{D_{2n-2}}\frac{\partial}{\partial x}K^{(2n-2)}(z;x,t) = \frac{d}{dx}h^{(2n-2)}(z;x,t)\Big|_{(x,t)=(x_b,t_b)}
\end{aligned} \tag{37}$$

for any $z \in \mathbb{C}$. Thus, the first equation in (24) is proven.

We now turn to the second equation in (24). Similarly to (35), we represent $\frac{\partial}{\partial t}K^{(2n)}(z)$ as

$$\frac{\partial}{\partial t}K^{(2n)}(z) = \frac{(-1)^{n-1}}{(z-\alpha_*)(z-\bar\alpha_*)}\times$$

$$
\begin{vmatrix}
\oint_{\hat{\gamma}_{m,1}} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,1}} \frac{\zeta d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,1}} \frac{\zeta^j d\zeta}{R_0(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,1}} \frac{[\zeta - \frac{1}{2}\sum_{j=0}^{4n+1}\alpha_j]\zeta^{2n-2} d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,1}} \frac{(\zeta-\alpha_*)(\zeta-\bar{\alpha}_*) d\zeta}{(\zeta-z)R(\zeta)} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\oint_{\hat{\gamma}_{m,n-1}} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,n-1}} \frac{\zeta d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,n-1}} \frac{\zeta^j d\zeta}{R_0(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,2}} \frac{[\zeta - \frac{1}{2}\sum_{j=0}^{4n+1}\alpha_j]\zeta^{2n-2} d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,n-1}} \frac{(\zeta-\alpha_*)(\zeta-\bar{\alpha}_*) d\zeta}{(\zeta-z)R(\zeta)} \\
\oint_{\hat{\gamma}_{c,1}} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,1}} \frac{\zeta d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,1}} \frac{z^j d\zeta}{R_0(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,1}} \frac{\zeta^{2n-2} d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,1}} \frac{(\zeta-\alpha_*)(\zeta-\bar{\alpha}_*) d\zeta}{(\zeta-z)R(\zeta)} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\oint_{\hat{\gamma}_{c,n-1}} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,2}} \frac{\zeta d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,n-1}} \frac{\zeta^j d\zeta}{R_0(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,n-1}} \frac{[\zeta - \frac{1}{2}\sum_{j=0}^{4n+1}\alpha_j]\zeta^{2n-2} d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,n-1}} \frac{(\zeta-\alpha_*)(\zeta-\bar{\alpha}_*) d\zeta}{(\zeta-z)R(\zeta)} \\
\oint_{\hat{\gamma}_{m,n}} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,n}} \frac{\zeta d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,n}} \frac{\zeta^j d\zeta}{R_0(\zeta)} & \cdots & \oint_{\hat{\gamma}_{m,n}} \frac{[\zeta - \frac{1}{2}\sum_{j=0}^{4n+1}\alpha_j]\zeta^{2n-2} d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{m,n}} \frac{(\zeta-\alpha_*)(\zeta-\bar{\alpha}_*) d\zeta}{(\zeta-z)R(\zeta)} \\
\oint_{\hat{\gamma}_{c,n}} \frac{d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,n}} \frac{\zeta d\zeta}{R(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,n}} \frac{\zeta^j d\zeta}{R_0(\zeta)} & \cdots & \oint_{\hat{\gamma}_{c,n}} \frac{[\zeta - \frac{1}{2}\sum_{j=0}^{4n+1}\alpha_j]\zeta^{2n-2} d\zeta}{R(\zeta)} & \oint_{\hat{\gamma}_{c,n}} \frac{(\zeta-\alpha_*)(\zeta-\bar{\alpha}_*) d\zeta}{(\zeta-z)R(\zeta)}
\end{vmatrix} ,
$$

$$(38)$$

where $j = 0, 1, \cdots, 2n-4$ and $\alpha_* \in \mathbb{C}$ is arbitrary. Using the identity

$$
\left(\zeta - \frac{1}{2} \sum_{j=0}^{4n-3} \alpha_j \right) (\zeta - \alpha^*)(\zeta - \bar{\alpha}^*)\zeta^{2n-4} = \zeta^{2n-1} - \frac{\zeta^{2n-2}}{2} \sum_{j=0}^{4n+1} \alpha_j + O(\zeta^{2n-3}) , \qquad (39)
$$

where $\alpha^* = \frac{1}{2}(\alpha_{4n-2} + \alpha_{4n})$, we can reduce the integrand in the $(2n-1)$st (next to the last) column of the latter determinant to

$$
\frac{\left[\zeta - \frac{1}{2}\sum_{j=0}^{4n-3}\alpha_j \right](\zeta - \alpha^*)(\zeta - \bar{\alpha}^*)\zeta^{2n-4}}{R(\zeta)} = \frac{\left[\zeta - \frac{1}{2}\sum_{j=0}^{4n-3}\alpha_j \right]\zeta^{2n-4}}{R_0(\zeta)} + O(\delta) \qquad (40)
$$

as $\delta \to 0$. The latter estimate is valid if $\zeta \neq \alpha$, $\zeta \neq \bar{\alpha}$ uniformly on compact subsets of $\mathbb{C} \setminus \{\alpha, \bar{\alpha}\}$. Thus the integrand in all but the last integral in the $(2n-1)$st column can be replaced by $\frac{[\zeta - \frac{1}{2}\sum_{j=0}^{4n-3}\alpha_j]\zeta^{2n-4}}{R_0(\zeta)}$ with accuracy $O(\delta)$. We also note that the last integral in this column is bounded. Denoting the latter determinant by $\hat{K}$ and applying to it the same arguments as we applied to (35), and also using (40), we obtain

$$
\hat{K} = \frac{1}{2} D_2 \frac{\partial}{\partial t} K^{(2n-2)}(z; x_b, t_b) . \qquad (41)
$$

Then (20), (30) and (41) yield

$$
\frac{d}{dt} h^{(2n)}(z; x, t) \bigg|_{(x,t)=(x_b,t_b)} = \frac{R_0(z)}{D_{2n-2}} \frac{\partial}{\partial t} K^{(2n-2)}(z; x, t) \bigg|_{(x,t)=(x_b,t_b)} = \frac{d}{dt} h^{(2n-2)}(z; x, t) \big|_{(x,t)=(x_b,t_b)}. \qquad (42)
$$

In the remaining case $n = 1$, expressions (35) and (38) become

$$
h_x^{(2)}(z; x_b, t_b) = -\lim_{\delta \to 0} \frac{R_0(z)}{D_2} \left[D_2 + \begin{vmatrix} \oint_{\hat{\gamma}_m} \frac{d\zeta}{(\zeta-z)R_0(\zeta)} & \oint_{\hat{\gamma}_m} \frac{d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}_c} \frac{d\zeta}{(\zeta-z)R_0(\zeta)} & \oint_{\hat{\gamma}_c} \frac{d\zeta}{R(\zeta)} \end{vmatrix} \right] = -R_0(z)
$$

$$
h_t^{(2)}(z; x_b, t_b) = 2\lim_{\delta \to 0} \frac{R_0(z)}{D_2} \left[\begin{vmatrix} \oint_{\hat{\gamma}_m} \frac{d\zeta}{(\zeta-z)R_0(\zeta)} & \oint_{\hat{\gamma}_m} \frac{\zeta d\zeta}{R(\zeta)} \\ \oint_{\hat{\gamma}_c} \frac{d\zeta}{(\zeta-z)R_0(\zeta)} & \oint_{\hat{\gamma}_c} \frac{\zeta d\zeta}{R(\zeta)} \end{vmatrix} - (z - 2a_0)D_2 \right] = -2(z - 2a_0)R_0(z)
$$

$$,$$
$$(43)$$

where $a_0 = \Re\alpha_0$. According to Corollary 4.4 from [9], in the genus zero region

$$h_x^{(0)}(z) = -R_0(z) \quad \text{and} \quad h_t^{(0)}(z) = -2(z + a_0)R_0(z) . \tag{44}$$

These expressions, combined with (43), complete the proof of the theorem for $n = 1$. $\quad\square$

4   Regular continuation principle

To prove the regular continuation principle, we need Theorem 3.1 and certain facts about the geometry of breaking curves. Namely, we need to prove that any regular nondegenerate breaking point lies on a smooth breaking curve and that any regular degenerate breaking point is an isolated point in the (x, t) plane.

Theorem 4.1. *If (x_b, t_b) is a regular nondegenerate breaking point, then there exists a breaking curve l passing through (x_b, t_b). Moreover, l is smooth and defined uniquely.*

Proof. If (x_b, t_b) is a regular nondegenerate breaking point, then $\exists z_0 \in \gamma$, such that $h'(z_0)$ but $h''(z_0) \neq 0$. Thus, z_0 and (x_b, t_b) satisfy the system

$$\begin{cases} h'(z; x, t) & = 0 \\ \Im h(z; x, t) & = 0 \end{cases} \tag{45}$$

of three real equation for four real variables u, vx, t, where $z = u + iv$.

According to Theorem 4.5 below, if $z \notin \mathbb{R}$ and if z is not a branchpoint, then $\Im h_x(z)$ and $\Im h_t(z)$ cannot be zero simultaneously. Let us assume, for example, that $\Im h_t(z_0) \neq 0$. Then, using the Cauchy-Riemann equations and the fact that $h'(z_0) = 0$, the Jacobian of the system (45) at (z_0, x_b, t_b) is

$$\begin{vmatrix} \frac{\partial}{\partial u}\Re h' & \frac{\partial}{\partial v}\Re h' & \frac{\partial}{\partial t}\Re h' \\ \frac{\partial}{\partial u}\Im h' & \frac{\partial}{\partial v}\Im h' & \frac{\partial}{\partial t}\Im h' \\ \frac{\partial}{\partial u}\Im h & \frac{\partial}{\partial v}\Im h & \frac{\partial}{\partial t}\Im h \end{vmatrix} = |h''(z_0)|^2 \cdot \Im h_t(z_0) \neq 0 . \tag{46}$$

Now, the Implicit Function Theorem completes the proof. $\quad\square$

Corollary 4.2. *Let (x_b, t_b) be a regular nondegenerate breaking point and z_0 be the corresponding (double) breaking point in the spectral plane. Then there exists a unique smooth curve λ, so that z_0 varies along λ as the correspondin breaking point (x_b, t_b) varies along l.*

To prove Theorem 4.4, we first need the following lemma.

Lemma 4.3. *If $\mathcal{R}$ is an hyperelliptic Riemann surface of genus $g > 0$ and if P_0, P_1 are two fixed points on $\mathcal{R}$, then there exists a holomorphic differential ω on $\mathcal{R}$ such that $\int_{P_0}^{P_1} \omega \neq 0$. Here we assume that the integral is single-valued, i.e., the contour of integration does not cross any α or β cycle of $\mathcal{R}$.*

Proof. Suppose the converse is true. Then for P_0 and P_1 the Abel map is trivial. By Abel's Theorem, $P_1 - P_0$ is a principle divisor, i.e., there exists a meromorphic function ϕ on $\mathcal{R}$ with the only pole at P_0 and the only zero at P_1, both the pole and the zero are simple. Then ϕ provides a diffeomorphism between $\mathcal{R}$ and the Riemann sphere, which is a contradiction to the fact that $g > 0$. $\quad\square$

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Theorem 4.4. *Let $h(z)$ be defined by (11) with some $N = 2n$, $n \in \mathbb{N}$. If z is not a branchpoint α_j, $j = 0, 1, \cdots, 4n + 1$, then*

$$|h_x(z)| + |h_t(z)| \neq 0. \tag{47}$$

Proof. Let us fix some z. In the case $n = 0$, (47) follows from (44). In the case $n > 0$, according to (18), $|h_x(z)| + |h_t(z)| = 0$ is equivalent to

$$\left| \frac{\partial}{\partial x} K(z) \right| + \left| \frac{\partial}{\partial t} K(z) \right| = 0 . \tag{48}$$

Let us assume that (48) is true. Consider $\frac{\partial}{\partial x} K(z)$, $\frac{\partial}{\partial t} K(z)$ given by (23). If the period vector $Q(z) =$

$$\mathrm{Col} \left(\oint_{\Gamma_{m,1}} \frac{d\zeta}{(\zeta - z) R(\zeta)}, \cdots, \oint_{\Gamma_{m,n}} \frac{d\zeta}{(\zeta - z) R(\zeta)}, \oint_{\Gamma_{c,1}} \frac{d\zeta}{(\zeta - z) R(\zeta)}, \cdots, \oint_{\Gamma_{m,n}} \frac{d\zeta}{(\zeta - z) R(\zeta)} \right) \tag{49}$$

of the meromorphic differential $\eta = \frac{d\zeta}{(\zeta - z) R(\zeta)}$ on the Riemann surface $\mathcal{R}$ is different from zero, then $Q(z)$ is a nontrivial linear combination of columns K_j and their complex conjugates from the determinant $\frac{\partial}{\partial x} K(z)$. Substituting this linear combination into $\frac{\partial}{\partial t} K(z)$, we see that, according to (48), a nontrivial linear combination of columns of determinant D, given by (21), is zero. Since $D \neq 0$, the obtained contradiction shows that (47) is true. To complete the proof, it remains to show that $Q(z) \neq 0$.

Note that η is an abelian differential of the third kind $\eta = \frac{d\zeta}{(\zeta - z) R(\zeta)}$ (a meromorphic differential with nonzero residues) on $\mathcal{R}$. Riemann bilinear relation for η is (see, for example, [2]),

$$\sum_{k=1}^{N} (A'_k B_K - A_k B'_k) = 2\pi i \sum c_j \int_{P_0}^{P_j} \omega , \tag{50}$$

where: ω is an arbitrary holomorphic differential on $\mathcal{R}$ with α and β periods $\{A'_k, B'_k\}$ respectively; $\{A_k, B_k\}$ are α and β periods of η respectively; P_0 is an arbitrary point on $\mathcal{R}$; P_j are the poles of η in $\mathcal{R}$ and c_j are their residues; the summation in the right hand side of (50) is taken over all the poles; a single-valued branch of the (multi-valued) integral $\int_{P_0}^{P_j} \omega$ is taken in the right hand side of (50), i.e., integration contours do not cross any main or any complementary arc except of $\gamma_{m,0}$ (that has endpoints α_0 and $\bar{\alpha}_0$). Since z is not a branchpoint, η has two simple poles at $P_1 = z$ on the main sheet and $P_2 = z$ on the secondary sheet of $\mathcal{R}$ with the residues $c_1 = \frac{1}{R(z)}$ and $c_2 = -\frac{1}{R(z)}$. Choosing $P_0 = \alpha_0$ and using the fact that all the α and β periods of η are zero, we can rewrite (50) as

$$\frac{1}{R(z)} \left[\int_{P_0}^{P_1} \omega + \int_{P_2}^{P_0} \omega \right] = 0 \tag{51}$$

Since on the secondary sheet $\omega(\zeta) = -\omega(\tilde{\zeta})$, where $\tilde{\zeta}$ is the projection of ζ on the main sheet, equation (51) becomes

$$\int_{\alpha_0}^{z} \omega = 0 , \tag{52}$$

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where the contour of integration lies on the main sheet. Note that (52) holds for all the basic holomorphic differentials of $\mathcal{R}$. However, this is contradicts Lemma 4.3. The proof is completed. $\qquad\square$

The proof of Theorem 4.4 can be slightly adjusted for the following statement.

Theorem 4.5. *Let $h(z)$ be defined by (11) with some $N = 2n$, $n \in \mathbb{N}$. If $z \notin \mathbb{R}$ and z is not a branchpoint α_j, $j = 0, 1, \cdots, 4n+1$, then*

$$|\Im h_x(z)| + |\Im h_t(z)| \neq 0. \tag{53}$$

Proof. Let us fix some z. In the case $n = 0$ (53) follows from (44). Consider the case $n > 0$. Since h_x, h_t are Schwarz symmetrical, we have

$$\Im h_x(z) = -\frac{1}{2}i\left[h_x(z) - h_x(\bar{z})\right], \quad \Im h_t(z) = -\frac{1}{2}i\left[h_t(z) - h_t(\bar{z})\right] . \tag{54}$$

Then $\Im h_x, t(z)$ are given by (18) and (23), where the last column $Q(z)$ of the periods of the meromorphic differential $\eta = \frac{d\zeta}{(\zeta-z)R(\zeta)}$ in (23) is replaced by the column $\tilde{Q}(z, \bar{z})$ of the periods of the meromorphic differential

$$-\frac{i}{2}\tilde{\eta} = -\frac{i}{2}\left[\frac{R(z)d\zeta}{(\zeta - z)R(\zeta)} - \frac{R(\bar{z})d\zeta}{(\zeta - \bar{z})R(\zeta)}\right] . \tag{55}$$

Following the arguments of Theorem 4.4, it is sufficeint to prove that the vector $\tilde{Q}(z, \bar{z})$ is not equal to zero for any $z \notin \mathbb{R}$, which is also not a branchpoint.

Assume that for some z, satisfying the requirements of the theorem, $\tilde{Q}(z, \bar{z}) = 0$. Since $\tilde{\eta}$ is an abelian differential of the third kind on $\mathcal{R}$, the right hand side of (50) is zero for any holomorphic differential ω. The differential $\tilde{\eta}$ has simple poles at $\zeta = z$ and $\zeta = \bar{z}$ with residues ± 1 respectively. So, the contribution of these two poles to the right hand side of (50) is $2\pi i \int_{\bar{z}}^{z} \omega$. The remaining two poles $\zeta = z$ and $\zeta = \bar{z}$ on the second sheet of $\mathcal{R}$ give exactly the same contribution. Thus, the Riemann bilinear relation implies

$$\int_{\bar{z}}^{z} \omega = 0 \tag{56}$$

for all the holomorphic differentials on $\mathcal{R}$. The obtained contradiction with Lemma 4.3 completes the proof. $\qquad\square$

Let z_0 be the breaking point on the spectral plane that corresponds to a regular breaking point (x_b, t_b). If (x_b, t_b) is a degenerate breaking point then, according to (6), the degree of z_0 is greater than two, so that $h''(z_0) = 0$.

Theorem 4.6. *A regular degenerate breaking point (x_b, t_b) is an isolated point in the x, t-plane, that is, there exists a neighborhood of (x_b, t_b) that is free of other degenerated breaking points.*

Proof. A) Let us first consider the case when z_0 is not a branchpoint. Then there exists some $m = 3, 4, \cdots$, such that $h^{(k)}(z_0) = 0$, $k = 1, 2, \cdots, m - 1$, but $h^{(m)}(z_0) \neq 0$, so that z_0 and (x_b, t_b) satisfy the system of $2m - 1$ real equations

$$\begin{cases} h^{(k)}(z; x, t) &= 0, \quad k = 1, 2, \cdots, m - 1 \\ \Im h(z; x, t) &= 0 \end{cases} \tag{57}$$

for four real variables u, v, x, t, where $z = u + iv$. Consider the subsystem

$$\begin{cases} h^{(m-1)}(z; x, t) &= 0, \\ h'(z; x, t) &= 0 \ , \\ \Im h(z; x, t) &= 0 \end{cases} \tag{58}$$

of (57), which has a Jacoby matrix

$$\begin{pmatrix} \frac{\partial}{\partial u} \Re h^{(m-1)} & \frac{\partial}{\partial v} \Re h^{(m-1)} & \frac{\partial}{\partial x} \Re h^{(m-1)} & \frac{\partial}{\partial t} \Re h^{(m-1)} \\ \frac{\partial}{\partial u} \Im h^{(m-1)} & \frac{\partial}{\partial v} \Im h^{(m-1)} & \frac{\partial}{\partial x} \Im h^{(m-1)} & \frac{\partial}{\partial t} \Im h^{(m-1)} \\ \frac{\partial}{\partial u} \Re h' & \frac{\partial}{\partial v} \Re h' & \frac{\partial}{\partial x} \Re h' & \frac{\partial}{\partial t} \Re h' \\ \frac{\partial}{\partial u} \Im h' & \frac{\partial}{\partial v} \Im h' & \frac{\partial}{\partial x} \Im h' & \frac{\partial}{\partial t} \Im h' \\ \frac{\partial}{\partial u} \Im h & \frac{\partial}{\partial v} \Im h & \frac{\partial}{\partial t} \Im h & \frac{\partial}{\partial t} \Im h \end{pmatrix} . \tag{59}$$

is the Jacoby matrix of system (58). At the point $z = z_0$, similarly to (46), the 2×2 minor in the upper left corner of (59) is equal to $\left| h^{(m)}(z_0) \right|^2 \neq 0$, whereas the 3×2 block in the lower left corner is a zero matrix. Theorem 4.5 implies that the 3×2 block

$$\begin{pmatrix} \frac{\partial}{\partial x} \Re h' & \frac{\partial}{\partial t} \Re h' \\ \frac{\partial}{\partial x} \Im h' & \frac{\partial}{\partial t} \Im h' \\ \frac{\partial}{\partial t} \Im h & \frac{\partial}{\partial t} \Im h \end{pmatrix} \tag{60}$$

is of at least rank $\rho = 1$. According to the Implicit Function Theorem, it is sufficient to show that the latter block has rank $\rho = 2$ in order to prove the theorem. To complete the proof, we assume $\rho = 1$ and obtain a contradiction.

Let us first obtain a contradiction in the case when $h(z)$ at $z = z_0$ is given by (11) with $n = 0$. In this case $h_x(z)$ and $h_t(z)$ are given by (44), so that

$$h_{xz} = -\frac{z - a}{R(z)}, \quad h_{tz} = -2\frac{z^2 - a^2}{R(z)} - 2R(z), \tag{61}$$

where $R(z) = \sqrt{(z - \alpha)(z - \bar{\alpha})}$ and $\alpha = a + ib$. Since for arbitrary f and g

$$\begin{vmatrix} \Re f & \Re g \\ \Im f & \Im g \end{vmatrix} = \Im(\bar{f}g) , \tag{62}$$

the assumption $\rho = 1$ implies $\Im \bar{h}_{xz} h_{tz} = 0$. Direct calculation yields

$$\Im h_{xz}(z) h_{tz}(z) = \frac{2\Im z \left[2|z - a|^2 - b^2 \right]}{|z - \alpha|^2} . \tag{63}$$

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Since $\Im z_0 > 0$, the point z_0 must be on the upper semicircle

$$|z - a| = \frac{b}{\sqrt{2}}. \tag{64}$$

Now, let us show that

$$\begin{vmatrix} \Im h_x(z) & \Im h_t(z) \\ h_{xz}(z) & h_{tz}(z) \end{vmatrix} = \frac{2}{R(z)} \begin{vmatrix} \Im R(z) & (u + a)\Im R(z) + v\Re R(z) \\ z - a & (z - a)^2 + b^2 + z^2 - a^2 \end{vmatrix} \neq 0 \tag{65}$$

for any $z = u + iv$ with $u > 0$ satisfying (64). Substituting (64) into the determinant in the right hand side of (65) yield

$$(z - a)\left[3(u - a)\Im R(z) - v\Re R(z)\right] = (z - a)\Im\left[\left[3(u - a) - iv\right]R(z)\right] . \tag{66}$$

Moreover, (64) yields

$$R(z) = \sqrt{(z - a)^2 + b^2} = \sqrt{z - a}\sqrt{3(u - a) - iv} , \tag{67}$$

which, together with (66), yield

$$\begin{vmatrix} \Im h_x(z) & \Im h_t(z) \\ h_{xz}(z) & h_{tz}(z) \end{vmatrix} = \frac{2(z - a)}{R(z)}\Im\left[(3(u - a) - iv)^{\frac{3}{2}}((u - a) - iv)^{\frac{1}{2}}\right] \tag{68}$$

To prove $\rho = 2$, we need to prove

$$3\arg\left(3(u - a) - iv\right) + \arg\left((u - a) - iv\right) \neq 2\pi m \tag{69}$$

for any $m \in \mathbb{Z}$, where $\theta = \arg\left((u - a) - iv\right)$ varies between 0 and π. Equation (69) can be rewrited as

$$\phi(\theta) = \theta + 3\tan^{-1}\left(-\frac{\tan\theta}{3}\right) \neq 2\pi m \tag{70}$$

if $\theta \leq \frac{\pi}{2}$; if $\theta > \frac{\pi}{2}$, we need to subtract 3π from this expression. Notice that $\phi(0) = 0$, $\phi(\frac{\pi}{2}) = -\pi$ and $\phi(\frac{\pi}{2}) = -2\pi$ and $\phi(\theta)$ is monotonically decreasing since

$$\phi'(\theta) = -8\frac{\sin^2\theta}{8\cos^2\theta + 1} < 0 . \tag{71}$$

So, inequality (70) holds for all $\theta \in (0, \pi)$. In the case $n = 0$, the proof is completed.

In the case of a positive genus $N = 2n$, derivatives h_x and h_t are given by (18), where D $\frac{\partial}{\partial x}K(z)$ and $\frac{\partial}{\partial t}K(z)$ are given by (21) and (23) respectively. Then

$$h_{xz}(z) = \frac{1}{D}\hat{K}_x(z), \quad h_{tz}(z) = \frac{1}{D}\hat{K}_t(z), \tag{72}$$

where $\hat{K}_x(z), \hat{K}_t(z)$ are obtained from determinants (23) respectively by replacing the last column $Q(z)$ with the column $\frac{d}{dz}\left(R(z)Q(z)\right)$ (in $\hat{K}_x(z), \hat{K}_t(z)$ the subscript does not mean differentiation).

Let us first prove that $h_{xz}(z), h_{tz}(z)$ cannot be zero simultaneously for any $z \in \mathcal{R}$. If vector $\frac{d}{dz}(R(z)Q(z)) \neq 0$, the proof is the same as for h_x, h_t in Theorem 4.4. In the case vector $\frac{d}{dz}(R(z)Q(z)) = 0$, we consider differential $\eta = \frac{d}{dz}\left(\frac{R(z)}{\zeta - z}\right)\frac{d\zeta}{R(\zeta)}$ on $\mathcal{R}$. Second order poles at $\zeta = z$ on the main and secondary sheets of $\mathcal{R}$ are the only poles of η. It is an abelian differential of the second kind since its residues are zeroes. Riemann bilinear relation for η and an arbitrary meromorphic differential ω on $\mathcal{R}$ is given by (see, for example, [?])

$$\sum_{k=1}^{N}(A_k'B_K - A_kB_k') = 2\pi i \sum_{P} \text{Res } u\omega \,, \tag{73}$$

where the summation is taken over all the poles P of the meromorphic function $u = \int \eta$ and of the meromorphic differential ω. Here $\{A_k', B_k'\}$ are α and β periods of ω respectively and $\{A_k, B_k\}$ are α and β periods of η respectively. Take ω to be a holomorphic differential. Since all the periods of η are zero and residues of $u\omega$ at z are the same on the both sheets of $\mathcal{R}$, we can reduce (73) to

$$\text{Res } (u\omega)|_{\zeta=z} = 0 \,, \tag{74}$$

where z is on the main sheet. Since $\text{Res } u|_{\zeta=z} = 1$ and ω is any holomorhic differential, we obtain a contradiction. Thus, the second row in the determinant

$$\begin{vmatrix} \Im h_x(z) & \Im h_t(z) \\ h_{xz}(z) & h_{tz}(z) \end{vmatrix} \tag{75}$$

is not zero.

Since both rows of the determinant (75) are nonzero (for every $z \in \mathcal{R}$ that is not a branchpoint), it is sufficient to show that for any $\xi \in \mathbb{C}$ and any $z \in \mathcal{R}$, the vector

$$V(\xi, z) = (V_1(\xi, z), V_2(\xi, z)) = (\Im h_x(z) - \xi h_{xz}(z), \Im h_t(z) - \xi h_{tz}(z)) \tag{76}$$

is not zero. Components of $V(\xi, z)$ can be represented as

$$V_1(\xi, z) = \frac{1}{D}\hat{K}_1(\xi, z), \quad V_2(\xi, z) = \frac{1}{D}\hat{K}_2(\xi, z), \tag{77}$$

where determinants $\hat{K}_{1,2}(\xi, z)$ are obtained from determinants $\frac{\partial}{\partial x}K(z)$ and $\frac{\partial}{\partial t}K(z)$ in (23) respectively by replacing the last column $Q(z)$ with

$$Z(\xi, z) = -\frac{i}{2}[R(z)Q(z) - R(\bar{z})Q(\bar{z})] - \xi\frac{d}{dz}(R(z)Q(z)) \,. \tag{78}$$

If the vector $Z(\xi, z) \neq 0$ then, as in the proof of Theorem 4.4, we can establish that $V_1(\xi, z)$ and $V_2(\xi, z)$ cannot be zero simultaneously. In the remaining case $Z(\xi, z) = 0$ we consider the meromorphic differential

$$\eta = \left\{-\frac{i}{2}\left(\frac{R(z)}{\zeta - z} - \frac{R(\bar{z})}{\zeta - \bar{z}}\right) - \xi\frac{d}{dz}\left(\frac{R(z)}{\zeta - z}\right)\right\}\frac{d\zeta}{R(\zeta)} \tag{79}$$

on $\mathcal{R}$. This is an abelian differential of the third kind with poles at $\zeta = z$ and $\zeta = \bar{z}$ on the both sheets of $\mathcal{R}$. The residues of η at $\zeta = z$ and $\zeta = \bar{z}$ (on the main sheet) are $-\frac{i}{2}$ and $\frac{i}{2}$

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respectively. Thus, we can repeat the arguments of Theorem 4.4 to prove that $Z(\xi, z) = 0$ is not possible.

B) Let us now consider the case when z_0 is a branchpoint. If $h(z)$ at $z = z_0$ is given by (11) with $n = 0$ (genus zero case), the statement of the theorem was proven in [10], Lemma 3.21. Otherwise, we assume $n > 0$. Note that if z_0 is a branchpoint, say, $z_0 = \alpha_{2j}$, then $h(z; x_b, t_b) = (z - z_0)^{m+\frac{1}{2}}[M + O(z - z_0)]$ in a vicinity of z_0, where $M \neq 0$ and $m = 2, 3, \cdots$. Therefore, the branchpoints α_{2k} satisfy the system

$$\begin{cases} K(\alpha_{2k}) & = 0, \quad k = 0, 1, \cdots, 2n, \\ K^{(l)}(\alpha_{2j}) & = 0, \quad l = 1, 2, \cdots, m-1, \end{cases} \tag{80}$$

which is the system of modulation equations (4) for the branchpoints in the upper halfplane with the requirement of additional degeneracy at α_{2j}. According to [8], we can use $K(z)$ given by (22). As in part A), consider the subsystem

$$\begin{cases} K(\alpha_{2k}) & = 0, \quad k = 0, 1, \cdots, j-1, j+1, \cdots, 2n, \\ K^{(m-1)}(\alpha_{2j}) & = 0, \\ K(\alpha_{2j}) & = 0 \end{cases} \tag{81}$$

of (80), which is a system of $2n+2$ complex equations for $2n+1$ complex variables $\alpha_{2k}, k = 0, 1, \cdots, 2n$, and two real variables x, t. As it was shown in [8], the Jacobian matrix of the first $2n$ equations with respect to the variables $\alpha_{2k}, k = 0, 1, \cdots, j-1, j+1, \cdots, 2n$ is diagonal and invertible. Since $M \neq 0$, one can show that, similarly to [8], $\frac{\partial}{\partial \alpha_{2j}} K^{(m-1)}(\alpha_{2j}) \neq 0$. So, in order to prove that the Jacobian of (81) is nonzero, it remains to show that

$$\begin{vmatrix} K_x(\alpha_{2j}) & K_t(\alpha_{2j}) \\ K_x(\bar{\alpha}_{2j}) & K_t(\bar{\alpha}_{2j}) \end{vmatrix} \neq 0 , \tag{82}$$

where the fact that $K(z)$ is Schwarz symmetrical was taken into account. Our arguments now are similar to those of part A). If vector $Q(\alpha_{2j}) \neq 0$ (see (23)), then the rows of the latter determinant are nonzero. Suppose $Q(\alpha_{2j}) = 0$. Consider the meromorphic differential $\eta = \frac{d\zeta}{(\zeta - \alpha_{2j})R(\zeta)}$, whoose only pole is $\zeta = \alpha_{2j}$. This is an abelian differential of the second kind with zero periods. So, it satisfies (74), where ω is an arbitrary abelian differntial, which cannot be true. Thus, the rows of (82) are nonzero.

To complete the proof, it is sufficient to show that for any $\xi \in \mathbb{C}$ the vector

$$W(\xi) = (W_1(\xi), W_2(\xi)) = \left(K_x(\alpha_{2j}) + \xi K_x(\bar{\alpha}_{2j}), K_t(\alpha_{2j}) + \xi K_(\bar{\alpha}_{2j}) \right) \tag{83}$$

is not zero. Components of $W(\xi)$ can be represented as

$$W_1(\xi) = \frac{1}{D}\tilde{K}_1(\xi), \quad W_2(\xi) = \frac{1}{D}\tilde{K}_2(\xi), \tag{84}$$

where determinants $\tilde{K}_{1,2}(\xi)$ are obtained from determinants $\frac{\partial}{\partial x}K(z)$ and $\frac{\partial}{\partial t}K(z)$ in (23) respectively by replacing the last column $Q(z)$ with $Y(\xi) = Q(\alpha_{2j}) - \xi Q(\bar{\alpha}_{2j})$. If vector

$Y(\xi) \neq 0$ then $W_1(\xi)$ and $V_2(\xi)$ cannot be zero simultaneously and the proof is completed. In the remaining case $Y(\xi) = 0$ we consider the meromorphic differential

$$\eta = \frac{d\zeta}{(\zeta - \alpha_{2j})R(\zeta)} + \xi \frac{d\zeta}{(\zeta - \bar{\alpha}_{2j})R(\zeta)} \tag{85}$$

on $\mathcal{R}$. This is an abelian differential of the second kind with poles at $\zeta = \alpha_{2j}$ and $\zeta = \bar{\alpha}_{2j}$. If vector $Y(\xi) = 0$ then all the periods of η are zero and, using (74) as above, we obtain a contradiction.

C) So far we considered only the case when at the breaking point (x_b, t_b) the topology of zero level curves of $\Im h(z; x, t)$ in the spectral plane changes only at one point z_0. In general, it is possible that the change of topology occurs at two (or more) points z_0 and z_1 simultaneously (note though that the same two branches of $\Im h(z; x, t) = 0$ cannot intersect more than one time). Assuming that both z_0 and z_1 are double points, we have two sets of equations (45) valid at $z = z_0$ and $z = z_1$ with the same $x = x_b, t = t_b$). Thus we have six real equations for six real unknowns which, according to (46), have a nonvanishing Jacobian. Thus, such breaking points (x_b, t_b) are isolated points on the x, t-plane. The proof of the theorem is completed. $\qquad \square$

We now use Theorem 3.1, as well as the results of this section, to prove the regular continuation principle in the case when all the branchpoints are bounded and stay away from the real axis.

Theorem 4.7. *Let the nonlinear steepest descent asymptotics for solution $q(x, t, \varepsilon)$ of the NLS (1) be valid at some point (x_b, t_b). If (x_*, t_*) is an arbitrary point, connected with (x_b, t_b) by a piecewise-smooth path Σ, if the countour $\gamma(x, t)$ of the RHP (2) does not interact with singularities of $f_0(z)$ as (x, t) varies from (x_b, t_b) to (x_*, t_*) along Σ, and if all the branchpoints are bounded and stay away from the real axis, then the nonlinear steepest descent asymptotics (with the proper choice of the genus) is also valid at (x_*, t_*).*

Proof. Let point (x_b, t_b) belongs to the genus $N = 2n$ region, $n \in \mathbb{N}$ of the solution $q(x, t, \varepsilon)$. If Σ does not intersect any breaking curve, or can be continuously deformed so that it does not intersect any breaking curve (while still satisfying the conditions of the theorem), the proof follows from the Evolution Theorem of [9]. Otherwise, suppose traversing Σ we find that at some $(x_b, t_b) \in \Sigma$ (breaking point) the inequalities (5) fail, say, at $z_0 \in \gamma_{c,j}$. According to Theorem 4.6, we can assume that: z_0 is the only breaking point in the (upper) spectral plane corresponding to (x_b, t_b), and; z_0 is a double (nondegenerate) breaking point. Otherwise (x_b, t_b) is a degenerate breaking point that can be avoided by a small deformation of Σ. Then, by Theorem 4.1, there is a breaking curve l passing through (x_b, t_b). If inequality (5) for the arc $\gamma_{c,j}$ fails only at one point (x_b, t_b) of the contour Σ, i.e., if it holds on Σ on a (punctured) vicinity of (x_b, t_b), then the breaking point (x_b, t_b) can be removed by a small variation of Σ. Otherwise, we can assume that Σ is transversal to l at (x_b, t_b). Then $D_\Sigma \Im h(z_0; x, t)|_{(x,t)=(x_b,t_b)} \leq 0$ (see Fig. 3), where D_Σ denotes the directional derivative along Σ. Moreover, according to Theorem 4.5,

$$D_\Sigma \Im h(z_0; x, t)|_{(x,t)=(x_b,t_b)} < 0. \tag{86}$$
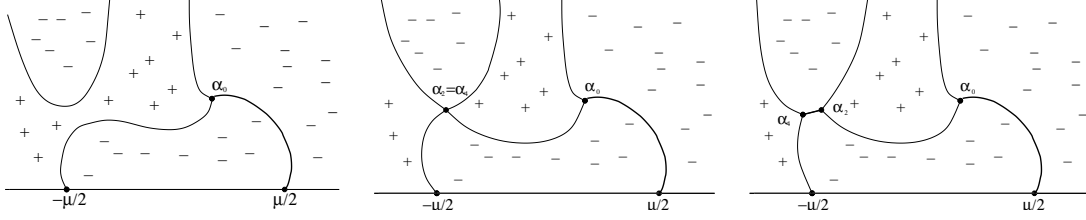
Figure 3: Transition from genus $N = 0$ to genus $N = 2$, where (5) for the complementary arc γ_c fails at $z_0 \in \gamma_c$ (center). Zero level curves and signs of $\Im h$ are shown: left, before the break, $N = 0$; center, at the break, $\Im h^{(}0) = \Im h^{(}2)$; right, after the break, $N = 2$. Note that $z_0 = \alpha_2 = \alpha_4$ at the break.

Let us plant two additional branchpoints $\alpha_{4n+2}, \alpha_{4n+4}$ at z_0, $2n = N$, which will open up a new main arc $\gamma_{m,n+1}$ as we move along Σ past the point (x_b, t_b). According to (11), $h(z; x, t) = h^{(N)}(z; x, t)$ has a different expression in the genus $N + 2$ region, i.e., beyond the point $(x_b, t_b) \in \Sigma$, which we denote by $h^{(N+2)}$. According to the Degeneracy Theorem from [9],

$$h^{(N+2)}(z; x_b, t_b) \equiv h^{(N)}(z; x_b, t_b) \ . \tag{87}$$

The nonlinear steepest descend method asymptotics will remain valid on Σ beyond the point $(x_b, t_b) \in \Sigma$ if the "newborn" main arc $\gamma_{m,n+1}$ would also satisfy (5), that is, if $\Im h^{(N+2)}(z; x_b, t_b) < 0$ to the left and to the right of $\gamma_{m,n+1}$. The latter inequality will be satisfied if

$$D_\Sigma h^{(N+2)}(z; x, t)|_{(z;x,t)=(z_0;x_b,t_b)} < 0. \tag{88}$$

But (88) follows from (86), where $h(z; x, t) = h^{(N)}(z; x, t)$ and Theorem 3.1. Thus, the nonlinear steepst descent asymptotics with te genus $2N + 2$ is valid on Σ beyond the breaking point (x_b, t_b). The case when one of the main arc inequalities of (5) is violated at (x_b, t_b) can be treated similarly. The case when a main or a complementary arc collapses to a point can be treated as above by moving in the opposite direction along Σ. So, we showed that the nonlinear steepest descent asymptotics is valid "automatically" as a breaking curve is crossed, which implies the theorem. $\qquad \square$

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